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Correlation for the calculation of turbulent friction in pipes

Correlación para el cálculo de la fricción turbulenta en tuberías

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Abstract

One of the essential parameters in hydraulic systems of pipe networks is the friction factor λ. The friction factor is determined using the implicit Colebrook-White equation through iterative methods, which makes its application challenging. In this work, a correlation based on the recursive method is developed to calculate the friction factor using the Colebrook-White equation. Two empirical relationships are proposed to finalise the correlation, with coefficients and exponents calibrated in Excel 2019. The results of the two proposed relationships were compared with the Swamee-Jain and Haaland relationships for recursive increments. For the λ_8 correlation, the maximum percentage error of the friction factor was 0,0000017%, for a relative roughness of 0,00001 and a Reynolds number of 4000. Additionally, the calculations yielded seven exact decimal digits for the friction factor. For Reynolds numbers greater than 4000, the percentage error decreases. As a result, it is concluded that the correlation based on the proposed explicit relationships satisfies the solution of the implicit Colebrook-White equation.

Keywords: Correlation, Colebrook-White equation, Percentage error, Friction factor, Recursive method

Resumen

En los sistemas hidráulicos de redes de tuberías, uno de los parámetros fundamentales es el factor de fricción λ. El factor de fricción se determina con la ecuación implícita de Colebrook-White por medios iterativos, lo cual dificulta su aplicación. En el presente trabajo se construye una correlación basada en el método recursivo para el cálculo del factor de fricción, para lo cual se empleó la ecuación de Colebrook-White. Para el cierre de la correlación se proponen dos relaciones empíricas, donde sus coeficientes y exponentes fueron calibrados en Excel 2019. Se compararon los resultados de las dos relaciones que se proponen con las relaciones de Swamee-Jain y Haaland, para incrementos recursivos, donde para la correlación λ_8 se obtuvo el error porcentual máximo del factor de fricción de 0,0000017 %, para la rugosidad relativa de 0,00001 y número de Reynolds 4000; así como, los decimales arrojaron siete dígitos decimales exactos para el factor de fricción. Para Reynolds mayores de 4000, el error porcentual disminuye. Se concluye que la correlación en función de las relaciones explícitas que se proponen satisface a la solución de la ecuación implícita de Colebrook-White.

Palabras clave: correlación, ecuación de Colebrook-White, error porcentual, factor de fricción, método recursivo

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1. Introduction

In pipe network systems in industrial processes, internal flow exhibits fluctuations in velocity, pressure, temperature, and other parameters. The flow is driven by pressure differentials, and flow friction is always present.

For a pipeline or measuring instrument, a gradient of pressure, velocity, and temperature occurs. The flow velocity is maximum in the central region, while at the wall, it is zero due to the no-slip condition. Therefore, a pressure decrease caused by viscous stresses represents an irreversible pressure loss known as pressure drop [1, 2]. There are abrupt pressure drops in the throat section in experimental flow measurement devices, such as the Venturi tube. In contrast, the pressure drops on the walls of the junctions between the throat section and the converging and diverging sections are lower compared to the central region of the flow [3].

The flow regime is classified into laminar, transitional, and turbulent. Smooth and parallel streamlines with orderly motion characterize the laminar flow. This flow is expected in fluids with high viscosity and low-speed motion. On the other hand, turbulent flow is characterized by random fluctuations of eddies at different scales.

These eddies transport mass, amount of motion, and energy to other flow regions, increasing the amount of motion, mass, and heat transfer in the affected areas. As a result, turbulent flow is associated with significant variations in the values of friction coefficients, mass transfer, and heat transfer [1, 2].

Osborne Reynolds [4] conducted experiments on pipe sections to study the flow and discovered that the flow regime is related to the ratio of inertial forces to viscous forces in the fluid. This ratio is known as the Reynolds number, R_e, and is calculated using the formula: $R_e = Vd/\nu$, where V is the average velocity, d is the internal diameter of the pipe, $\nu = \mu/\rho$ is the kinematic viscosity, μ is the dynamic viscosity, and ρ is the density of the fluid [1, 2].

The transition from laminar flow to turbulent flow is determined by the viscosity and velocity of the flow, as well as the pipe geometry, internal wall roughness, wall temperature, and other factors. In most practical conditions, flow is classified as follows: laminar flow for $R_e \leq 2300$, turbulent flow for $R_e \geq 4000$, and transitional flow in the range of $2300 \leq R_e \leq 4000$ [1, 2].

Colebrook and White [5, 6] proposed an implicit equation to calculate the friction factor λ, in turbulent flow in pipes, based on the results of their experimental research. This equation is known as the Colebrook-White equation (1).

$$\frac{1}{\sqrt{\lambda}} = -2log\left[\frac{\varepsilon}{3,7} + \frac{2,51}{R_e\sqrt{\lambda}}\right] \tag{1}$$

The Colebrook-White equation combines data for transitional and turbulent flow in smooth and rough pipes. The parameter ε represents the relative roughness and is defined as $\varepsilon = k/d$, where k is the average height of the material's roughness and d is the pipe's internal diameter. The parameter R_e is the Reynolds number. The parameters λ, ε and R_e are dimensionless.

The friction factor in the Colebrook-White equation cannot be solved accurately and explicitly using algebraic procedures. Therefore, the friction factor is determined using numerical methods through iterative procedures in computational codes, such as the Newton-Raphson method, bisection, fixed-point iteration, etc. This makes it challenging to obtain the friction factor in the design of pipeline networks due to the extensive and laborious calculations required by these iterative methods.

As an alternative solution to the implicit Colebrook-White equation, Moody [7] proposed using a graph that represents this equation. This graph is used in engineering to determine the friction factor. However, a numerical error is generated when determining the friction factor, which implies obtaining an approximate result.

The literature describes empirical correlations that provide an approximate solution for calculating the friction factor. These correlations are based on the Colebrook-White equation. Among the most well-known and widely used empirical correlations are the Swamee-Jain equation (2) [8], with a maximum estimated error of 3.2%, and the Haaland equation (3) [9], with a maximum estimated error of 2.1%.

$$\frac{1}{\sqrt{\lambda}} = -2log\left[\frac{\varepsilon}{3,7} + \frac{5,74}{R_e^{0,9}}\right] \tag{2}$$

$$\frac{1}{\sqrt{\lambda}} = -1,8log\left[\left(\frac{\varepsilon}{3,7}\right)^{1,11} + \frac{6,9}{R_e}\right] \tag{3}$$

Several authors have proposed various empirical and explicit relationships to decrease the numerical error associated with the friction factor in relation to ε and R_e. They have employed different methods to obtain a more accurate solution to achieve this.

Some authors, such as Mikata and Walczack [10], Rollmann and Spindler [11] and Biberg [12] apply the Lambert ω function. Serghides [13], Vatankhah [14], and Azizi *et al.* [15] obtain correlations through combinations of algebraic procedures. Chen [16], Schorle *et al.* [17], Zigrang and Sylvester [18], Sousa *et al.* [19], Romeo *et al.* [20] and Offor and Alabi [21] obtain correlations using the recursive method with modifications of constants and exponents. Santos *et al.* [22] and Alfaro *et al.* [23] conduct experimental evaluations to calculate the friction factor.

In addition to the aforementioned authors, Pérez *et al.* [24] present a list of forty-nine (49) explicit relationships for calculating the friction factor. This list starts with the Moody equation [7] and ends with the one proposed by Azizi *et al.* [15].

It is worth mentioning that some recent studies have conducted reviews of the errors associated with the friction factor in correlations reported in the literature [25]. These studies reveal that the relationship proposed by Praks and Brkić [26] has a maximum percentage error of 0,001204%, the relationship proposed by Serghides [13] yields an error of 0,00256%, the relationship proposed by Vantakhak [14] has an error of 0,005952%, and the relationship proposed by Romeo *et al.* [20] presents an error of 0,007468%. Lamri and Easa [27] apply the Lagrange inversion theorem and obtain an error of 0,002% for four terms.

Based on the results obtained by the mentioned authors, it can be inferred that the error produced by each empirical equation is due to the structure of the equation with the algebraic terms it comprises and the coefficients and exponents used.

The numerical precision in the number of decimal digits of the friction factor is related to the percentage relative error. Therefore, it is essential to calibrate the coefficients and exponents to create a new empirical correlation that has a simple structure as a mathematical model.

This work develops an explicit correlation using the recursive method to calculate the friction factor for turbulent flow in pipes. Additionally, this correlation is evaluated for four explicit relationships that calculate the friction factor for the initial approximation. The methodology is described in Section 2, the results obtained for the friction factor and percentage errors are presented in Section 3, and finally, the conclusions of the analysis are discussed in Section 4.

2. Materials and methods

2.1. Graphical representation of the correlation curve fit

Figure 1 displays a generic scheme with three curve trajectories. It illustrates the curve of an implicit analytic function $y = f(x, y)$. Additionally, it shows the curve of an explicit empirical function $y = h(x)$, which is offset from the curve of the analytic function. The segmented curve corresponds to the correlation, which is the recursive function $y_{n+1} = f(x, y_n)$. This function approaches the curve of the analytic function.

At a local reference point (x_o, y_o) for the empirical function $y_o = h(x_o)$, the data x_o is within the range from x_a to x_b (x-axis), and the output data y_o is within the range from y_a to y_b (y-axis). For the analytic function, when x_o, the output data is y_m, establishing the reference point (x_o, y_m). Similarly, for recursion, when x_o, is the input data, the result is y_{n+1}, defining the reference position (x_o, y_{n+1}). For a fixed point x_o, as the algebraic terms of the recursion y_{n+1} increase, starting from y_o, the dependent variable y_{n+1} approaches the fixed value y_m This implies that the numerical error gradually decreases until achieving numerical convergence $y_m = y_{n+1}$. Therefore, for different input values x_o, the recursion curve would overlap the analytic curve in the range $x_a \leq x_o \leq x_b$, and the output data would fall within the range $y_a \leq y_{n+1} \leq y_b$.

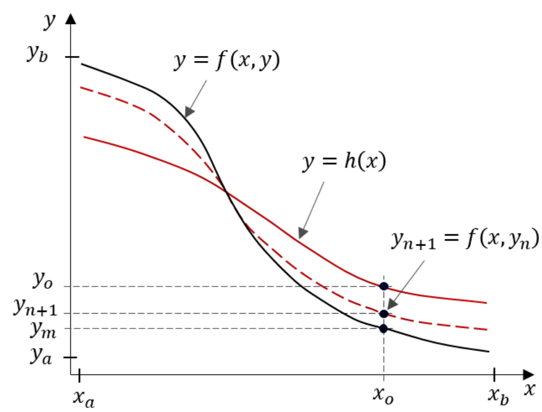

Figure 1. Basic schematic representation of the curves for the analytic, empirical, and recursive functions.

In Figure 1 the steps of the recursive method shown in equation (4).

$$\begin{aligned} y_1 &= f(x_o, y_o) \\ y_2 &= f(x_o, y_1) \\ y_3 &= f(x_o, y_2) \\ &\vdots \\ y_{n+1} &= f(x_o, y_n) \end{aligned} \tag{4}$$

Where the first approximation is y_1, the second is y_2, the third is y_3, and the last one y_{n+1}. The succession for y_{n+1}, increases progressively, starting from $n = 0$.

The function $y_o = h(x_o)$ is an explicit mathematical expression that provides a representation of a closed initial calculation, where y_o is the first approximate solution.

2.2. Explicit relationship

To calculate the initial first approximation of the friction factor λ_o, it is necessary to establish a mathematical expression as an explicit relationship for this approximate solution.

To obtain the explicit relationship λ_o, equation (1) of Colebrook-White [5,6] was considered. $\sqrt{\lambda}$ was removed from the argument of the Colebrook-White equation, and the positions of the coefficients a_i and exponents n_i were adjusted. As a result, the explicit relationship λ_o for the initial friction factor calculation was structured as equation (5).

$$\frac{1}{\sqrt{\lambda_o}} = a_1 log\left[\left(\frac{\varepsilon}{a_2}\right)^{n_1} + \left(\frac{a_3}{R_e}\right)^{n_2}\right] \quad (5)$$

The input data consists of the independent parameters: the relative roughness ε and the Reynolds number R_e. These parameters are set in the range of $1\times10^{-06} \leq \varepsilon \leq 0,05$ and $4000 \leq R_e \leq 1\times10^{+08}$. The output data is λ_o.

The coefficients a_1, a_2, a_3 and the exponents n_1 and n_2 of equation (5) were iteratively calibrated in an Excel 2019 spreadsheet using equation (1) of Colebrook-White as a reference. The best results regarding the magnitudes of the coefficients and exponents were selected to establish two explicit relationships.

Table 1 displays the magnitudes of the coefficients and exponents for the two proposed explicit relationships, presented as equations (6) and (7).

Table 1. Calibrated values of coefficients and exponents

	Coefficient			**Exponent**	
	$\mathbf{a_1}$	$\mathbf{a_2}$	$\mathbf{a_3}$	$\mathbf{n_1}$	$\mathbf{n_2}$
Eq. (6):	1,795	3,9	6,94	1,104	-
Eq. (7):	2	3,7	6,94	-	0,9

$$\frac{1}{\sqrt{\lambda_o}} = -a_1 log\left[\left(\frac{\varepsilon}{a_2}\right)^{n_1} + \frac{a_3}{R_e}\right] \quad (6)$$

$$\frac{1}{\sqrt{\lambda_o}} = -a_1 log\left[\frac{\varepsilon}{a_2} + \left(\frac{a_3}{R_e}\right)^{n_2}\right] \quad (7)$$

2.3. Correlation adjustment

Based on the recursive method, equation (1) of Colebrook-White [5, 6] was adjusted to calculate the friction factor using λ_{n+1}, as the output data and λ_n as the input data. The modified equation (8) is expressed as follows.

$$\frac{1}{\sqrt{\lambda_{n+1}}} = a\ log\left[b + c\frac{1}{\sqrt{\lambda_n}}\right] \quad (8)$$

Where $a = -2$, $b = \varepsilon/3,7$ y $c = 2,51/R_e$.

Equation (8) is based on the base 10 logarithm. This equation is used to perform the calculations in this work. It can also be expressed in terms of the natural logarithm as follows: $1/\sqrt{\lambda_{n+1}} = a_1 ln\left[b + c/\sqrt{\lambda_n}\right]$, where $a_1 = a/ln(10)$.

Using equation (8), the correlation expressed as equation (9) was established to calculate the increment of the succession $\lambda_{n+1} = \lambda_2, \lambda_4, \lambda_6, \lambda_8$.

$$\begin{aligned} \tfrac{1}{\sqrt{\lambda_8}} &= a\ log\left[b + ac\ log\left[b + cD\right]\right] \\ D &= a\ log\left[b + ac\ log\left[b + cC\right]\right] \\ C &= a\ log\left[b + ac\ log\left[b + cB\right]\right] \\ B &= a\ log\left[b + ac\ log\left[b + cA\right]\right] \end{aligned} \quad (9)$$

Where $D = \frac{1}{\sqrt{\lambda_6}}$, $C = \frac{1}{\sqrt{\lambda_4}}$, $B = \frac{1}{\sqrt{\lambda_2}}$ and $A = \frac{1}{\sqrt{\lambda_o}}$.

Equation (9) was extended by adding terms to evaluate up to $\lambda_{n+1} = \lambda_{20}$, although the mathematical expression for λ_{20} is not presented because the procedure is similar. The same principle as in equation (8) is applied.

The purpose of evaluating the correlation in a segmented manner was to determine the decrease in the percentage error of the friction factor for different values of the Reynolds number and relative roughness.

As input value of λ_o for $A = 1/\sqrt{\lambda_o}$ en in equation (9), four explicit relationships were considered: the Swamee-Jain relationship [8] (equation (2)), the Haaland relationship [9] (equation (3)), and the equations (6) and (7) proposed in this work. Each relationship was evaluated separately in equation (9).

The percentage error of the friction factor $\lambda(\%)$ was calculated using the following the equation (10).

$$\lambda(\%) = 100\left|\frac{\lambda_m - \lambda_{n+1}}{\lambda_m}\right| \quad (10)$$

Where λ_m represents the friction factor of the exact solution of the Colebrook-White equation, and λ_{n+1} represents the friction factor obtained from equation (9).

It is worth mentioning that all numerical calculations and graphs were performed in an Excel 2019 spreadsheet.

3. Results and discussion

3.1. Correlation and explicit relationships for friction factor calculation

The correlation created using the recursive method is expressed by equation (9), where the parameter values are $a = -2$, $b = \varepsilon/3,7$, and $c = 2,51/R_e$. This correlation is a mathematical expression that models the trajectory of the friction factor curve, λ, in relation to the relative roughness, ε, and the Reynolds number, R_e. Considering the term $A = 1/\sqrt{\lambda_o}$, in which the explicit relationships of $1/\sqrt{\lambda_o}$ are substituted, acting as the correlation's finalizing factor.

The two explicit relationships proposed in this work to close equation (9) are equations (6) and (7), and they are expressed as follows.

It is worth mentioning that equation (9) has a simple structure for direct calculations, which improves the precision in reducing the error of the friction factor with each additional term in the recursion. Therefore, the exact decimal digits of the friction factor increase.

Below, we present the percentage errors of the friction factor yielded by equation (9) for the explicit relationships that serve as a closure for $A = 1/\sqrt{\lambda_o}$, equation (2) by Swamee-Jain, equation (3) by Haaland, and the proposed equations (6) and (7).

3.2. Percentage errors of the friction factor

Figure 2 shows the plots of the curve trajectories of the percentage errors of the friction factor λ_o, calculated by the initial first approximation for equations (2), (3), (6) and (7). These curve trajectories are essential to understand the effect of the coefficients and exponents on variable values of the relative roughness, ε, and the Reynolds number, R_e.

For the Reynolds number range of $4000 \leq R_e \leq 1 \times 10^{08}$ and the relative roughness range of $1 \times 10^{-06} \leq \varepsilon \leq 0,05$, equation (6) exhibits an estimated maximum percentage error of 2.1%, while equation (7) has an error of 3.1%. Swamee-Jain's equation (2) shows an estimated maximum error of 3.2%, and Haaland's equation (3) has an error of 2.1%. In certain areas, equations (2), (3), (6), and (7) exhibit errors of around 0.1% (Figure 2). It is worth mentioning that the figures only depict curve trajectories for the relative roughness range of $0,00001 \leq \varepsilon \leq 0,05$.

For $\varepsilon = 0,05$ (Figure 2a), equations (3) and (6) exhibit trajectories with a horizontal trend starting from the local position $R_e = 1 \times 10^{05}$. The curve of equation (7) overlaps with the curve of equation (2) and shows a straight-line trend with a negative slope. For $\varepsilon = 0,00001$ (Figure 2d), the fluctuations in the friction factor curves are greater compared to the other curves illustrated in Figure 2.

For hydraulically rough pipes ($\varepsilon = 0,05$), the trends differ much more from each other than in hydraulically smooth pipes ($\varepsilon = 0,00001$), especially in the case of equations (2) and (7) for the curve in Figure 2a.

The curves show that the coefficients and exponents of each relationship have a dominant effect that defines their own trajectory behavior.

It is worth mentioning that, in Figure 2 and other figures shown below, there are inflection points in the curves during the descending peaks for specific Reynolds numbers, which are not visible because the error output data (y-axis) are absolute values according to equation (10). Additionally, it is essential to note that the vertical axis is on a logarithmic scale base 10 to facilitate the analysis of the curve trajectories.

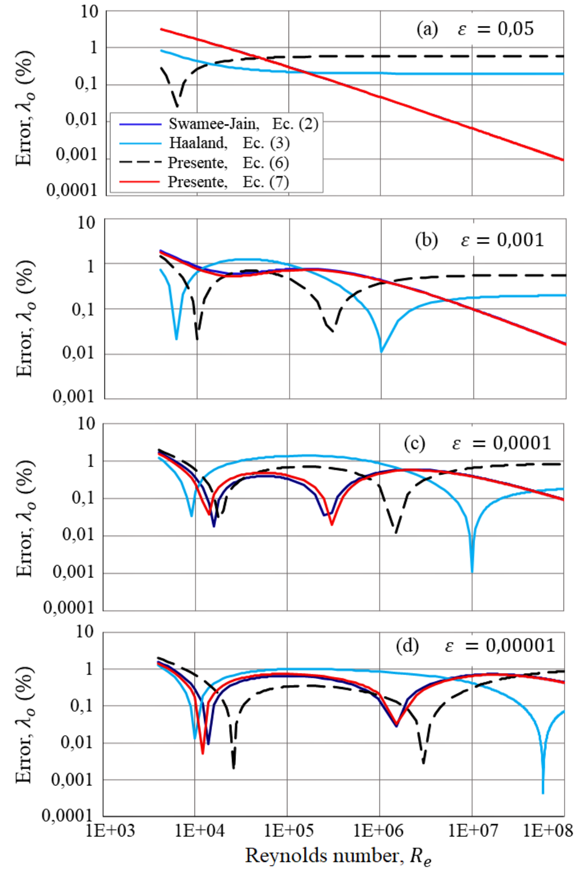

Figure 2. Percentage errors of the friction factor for λ_o, as the calculation of the first approximation of Equations (2), (3), (6) and (7)

Figures 3, 4, 5 and 6 show the trajectories of the curves for the percentage errors of the friction factor for λ_2, λ_4, λ_6 and λ_8. As the terms increase, equations (2), (3), (6) and (7) define their own curve trajectories. The curve trajectories exhibit the highest percentage relative error of the friction factor for each value of ε, at the position $R_e = 4000$, for λ_2, λ_4, λ_6 and λ_8

The regions with the highest errors of the friction factor are found at the local positions $\varepsilon = 0,00001$ and $R_e = 4000$, as illustrated in Figures 3d, 4d, 5d and 6d.

The equations (2), (3), (6) and (7) for λ_8 (Figure 6d) have errors less than 0,000002%. For λ_6 (Figure 5d), the maximum errors are 0,00006%. Forλ_4 (Figure 4d), the errors are 0,002%, and for λ_2 (Figure 3d), the errors are 0,061%.

The magnitudes of the coefficients and exponents in each explicit relationship exhibit a specific behavior, which influences the evolution of the curve trajectories as the relative roughness increases within the same range of Reynolds numbers, as illustrated in Figures 2 to 6. For future work, it is recommended to perform comparisons with other explicit relationships by substituting them into equation (9) to determine which one yields the lowest percentage errors.

It is worth mentioning that something similar to what is observed in Figure 2a also occurs in Figure 3a. For λ_2 and $\varepsilon = 0,05$, equations (2) and (7) show a more significant difference in hydraulically rough pipes, with a trend of straight lines with a negative slope. Similarly, equations (3) and (6) also exhibit straight trajectories and intersect in the region around $R_e = 1 \times 10^{05}$. This phenomenon is also observed in Figures 4a, 5a, and 6a, where the trajectories show a trend of straight lines for rough pipes ($\varepsilon = 0,05$), respectively.

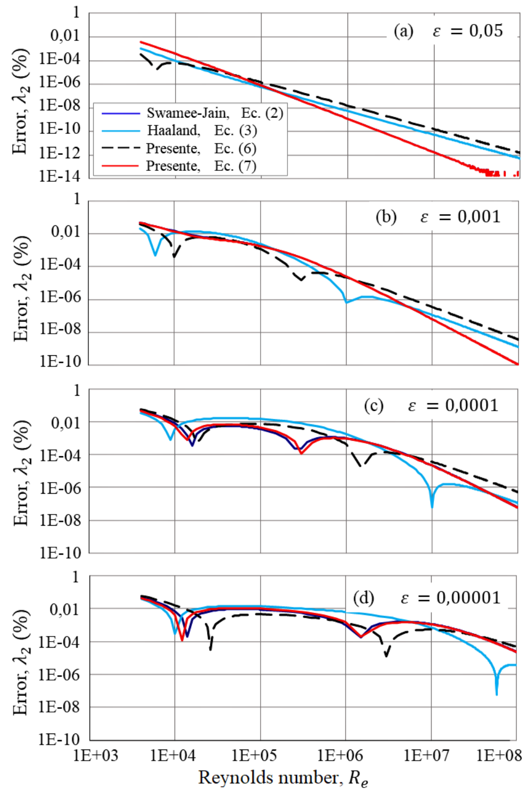

Figure 3. Percentage errors of the friction factor for λ_2

Table 2 displays the maximum errors of the friction factor for $\varepsilon = 0,00001$ and the local Reynolds numbers 4×10^{03}, 1×10^{04}, 1×10^{05}, 1×10^{06}, 1×10^{07} and 1×10^{08}. These values correspond to λ_o, λ_2, λ_4, λ_6 and λ_8, which are related to Figures 2, 3, 4, 5 and 6. For equation (6) for λ_8, considering the conditions $\varepsilon = 0,00001$ and $R_e = 4 \times 10^{03}$, a maximum percentage error of the friction factor of $1,7 \times 10^{-06}$% is obtained. For equations (2), (3) and (7), the errors are below $1,7 \times 10^{-06}$ %.

It is worth mentioning that as the recursion increases, the percentage errors decrease. Consequently, the friction factors for λ_8 exhibit seven exact decimal digits for $\varepsilon = 0,001$, eight exact decimal digits for $\varepsilon = 0,00001$ and nine exact decimal digits for $\varepsilon = 0,05$ and $\varepsilon = 0,0001$, compared to the friction factor of Colebrook-White equation (1), as shown in Table 3. For recursion values lower than λ_8, the exact decimal digits decrease. For λ_6 there are six exact decimal digits, for λ_4 there are five exact decimal digits, and for λ_2 there are four exact decimal digits.

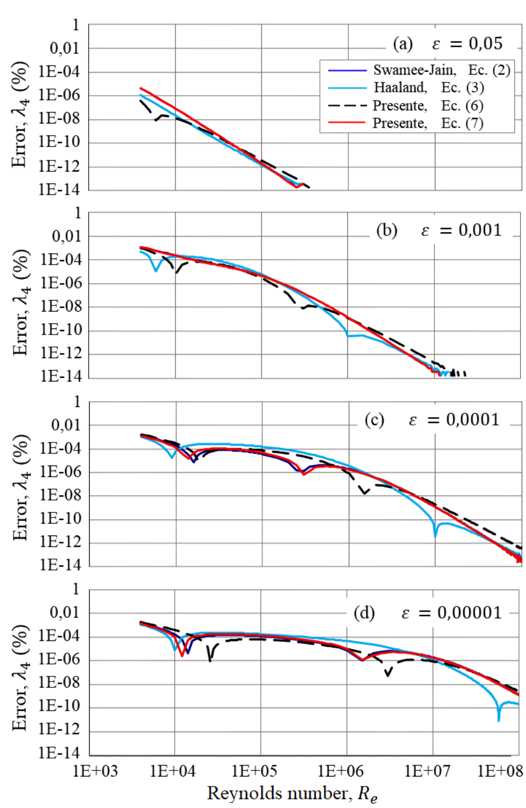

Figure 4. Percentage errors of the friction factor for λ_4

In equation (9) for λ_8, the maximum percentage error of the friction factor was obtained, with a value of 0.0000017%, which is significantly lower than other reported percentage errors. Brkić and Stajić [25], obtained errors around 0.001204%, Praks and Brkić [26] obtained an error of 0.002560%, and Serghides [13] obtained an error of 0.002560%. It is worth mentioning that the percentage errors reported by Brkić and Stajić [25] have not been verified by the authors of this study through numerical calculations. Therefore, they are presented solely for comparative purposes.

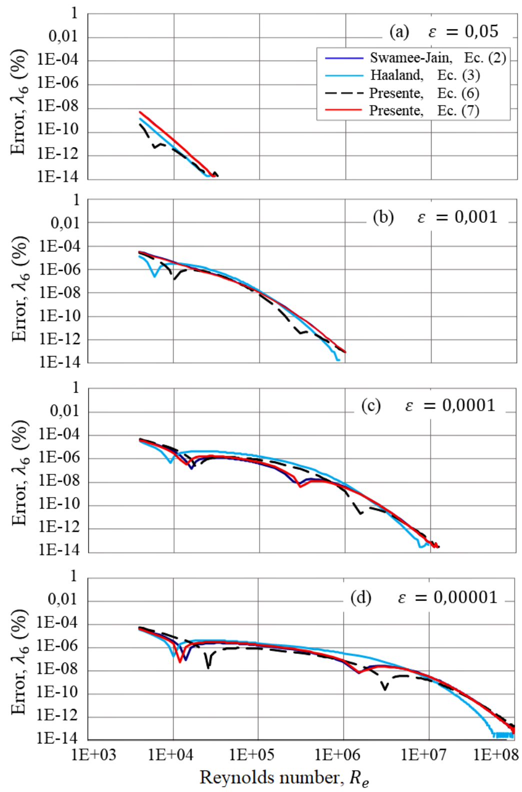

Figure 5. Percentage errors of the friction factor for λ_6

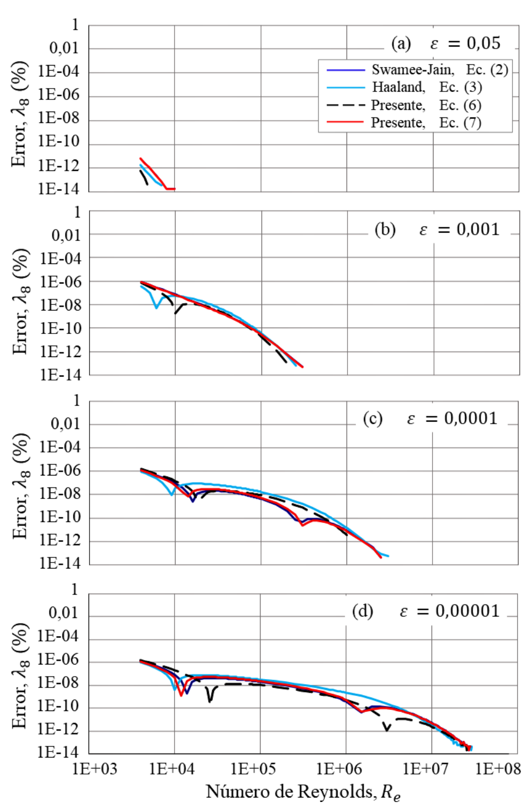

Figure 6. Percentage errors of the friction factor for λ_8

Table 2. Percentage errors, λ (%), of the friction factor for the recursions, for $\varepsilon = 0,00001$ and local R_e

Error	**Eq (2)**	**Eq (3)**	**Eq (6)**	**Eq (7)**
λ (%)	$R_e = 4E + 03$			
λ_0	1,6182	1,2790	2,0294	1,4803
λ_2	0,0482	0,0381	0,0610	0,0441
λ_4	0,0014	0,0011	0,0018	0,0013
λ_6	4,4E-05	3,5E-05	5,0E-05	4,0E-05
λ_8	1,4E-06	1,1E-06	1,7E-06	1,2E-06
λ (%)	$R_e = 1E + 04$			
λ_0	0,2958	0,0132	0,7047	0,1769
λ_2	0,0068	0,0003	0,0163	0,0041
λ_4	1,6E-04	7,2E-06	3,8E-04	9,6E-05
λ_6	3,7E-06	1,7E-07	8,8E-06	2,3E-06
λ_8	8,6E-08	3,9E-09	2,1E-07	5,2E-08
λ (%)	$R_e = 1E + 05$			
λ_0	0,6637	1,0164	0,3413	0,7522
λ_2	0,0088	0,0135	0,0045	0,0099
λ_4	1,2E-04	1,8E-04	6,0E-05	1,4E-04
λ_6	1,6E-06	2,4E-06	7,9E-07	1,8E-06
λ_8	2,1E-08	3,2E-08	1,1E-08	2,3E-08
λ (%)	$R_e = 1E + 06$			
λ_0	0,1380	0,8650	0,1948	0,2036
λ_2	0,0010	0,0062	0,0013	0,0014
λ_4	7,1E-06	4,5E-05	1,0E-05	1,1E-05
λ_6	5,1E-08	3,2E-07	7,2E-08	7,6E-08
λ_8	3,7E-10	2,3E-09	5,2E-10	5,4E-10
λ (%)	$R_e = 1E + 07$			
λ_0	0,6984	0,4194	0,3166	0,6651
λ_2	0,0011	7,0E-04	5,3E-04	0,0011
λ_4	2,0E-06	1,2E-06	8,8E-07	1,80E-06
λ_6	3,2E-09	1,9E-09	1,5E-09	3,1E-09
λ_8	5,3E-12	3,3E-12	2,4E-12	5,1E-12
λ (%)	$R_e = 1E + 08$			
λ_0	0,4423	0,0732	0,8823	0,4351
λ_2	2,4E-05	3,9E-06	4,7E-05	2,3E-05
λ_4	1,3E-09	2,1E-10	2,5E-09	1,3E-09
λ_6	6,4E-14	0	1,3E-13	6,4E-14
λ_8	0	0	0	0

Figure 7 displays the decrease in the percentage error of the friction factor as recursion increases for λ_2, λ_4, λ_6, λ_8, up to λ_{20}.

For equations (2), (3) (6) and (7), with, $\varepsilon = 0,00001$ and a local position of $R_e = 4E + 03$, the following percentage errors are obtained: λ_9, $2,85E - 07\%$, λ_{10}, $4,95E - 08$ %, λ_{11}, $8,55E - 09$ %; λ_{12}, $1,5E - 09$ %, λ_{14}, $4,5E - 11$ %, λ_{16}, $1,5E - 12\%$, λ_{18}, $5,5E - 14\%$, and λ_{20} 0,0%.

For other values of relative roughness and Reynolds numbers, the percentage errors of the friction factor are lower. This indicates that as the Reynolds number increases, fewer terms in the recursion are required to achieve numerical convergence.

Table 3. Comparison of numerical values of friction factors for the recursions considering the Colebrook-White equation $R_e = 4E + 03$ and local ε

	Eq (2)	Eq (3)	Eq (6)	Eq (7)
λ	Eq (1): $\lambda = 0,076986834$; $\varepsilon = 0,05$			
λ_0	0,0793827	0,0776348	0,0772007	0,0793531
λ_2	0,0769896	0,0769876	0,0769870	0,0769895
λ_4	0,0769868	0,0769868	0,0769868	0,0769868
λ_6	0,0769868	0,0769868	0,0769868	0,0769868
λ_8	0,0769868	0,0769868	0,0769868	0,0769868
λ	Ec. (1): $\lambda = 0,040910389$; $\varepsilon = 0,001$			
λ_0	0,0416954	0,0412161	0,0415108	0,0416423
λ_2	0,0409306	0,0409183	0,0409259	0,0409293
λ_4	0,0409109	0,0409105	0,0409107	0,0409108
λ_6	0,0409104	0,0409103	0,0409104	0,0409104
λ_8	0,0409103	0,0409103	0,0409103	0,0409103
λ	Ec. (1): $\lambda = 0,040008431$; $\varepsilon = 0,0001$			
λ_0	0,0406678	0,0404853	0,0407853	0,0406129
λ_2	0,0400278	0,0400224	0,0400312	0,0400262
λ_4	0,0400090	0,0400088	0,0400091	0,0400089
λ_6	0,0400084	0,0400084	0,0400084	0,0400084
λ_8	0,0400084	0,0400084	0,0400084	0,0400084
λ	Ec. (1): $\lambda = 0,039917166$; $\varepsilon = 0,00001$			
λ_0	0,0405631	0,0404277	0,0407272	0,0405080
λ_2	0,0399364	0,0399324	0,0399412	0,0399347
λ_4	0,0399177	0,0399176	0,0399178	0,0399176
λ_6	0,0399171	0,0399171	0,0399171	0,0399171
λ_8	0,0399171	0,0399171	0,0399171	0,0399171

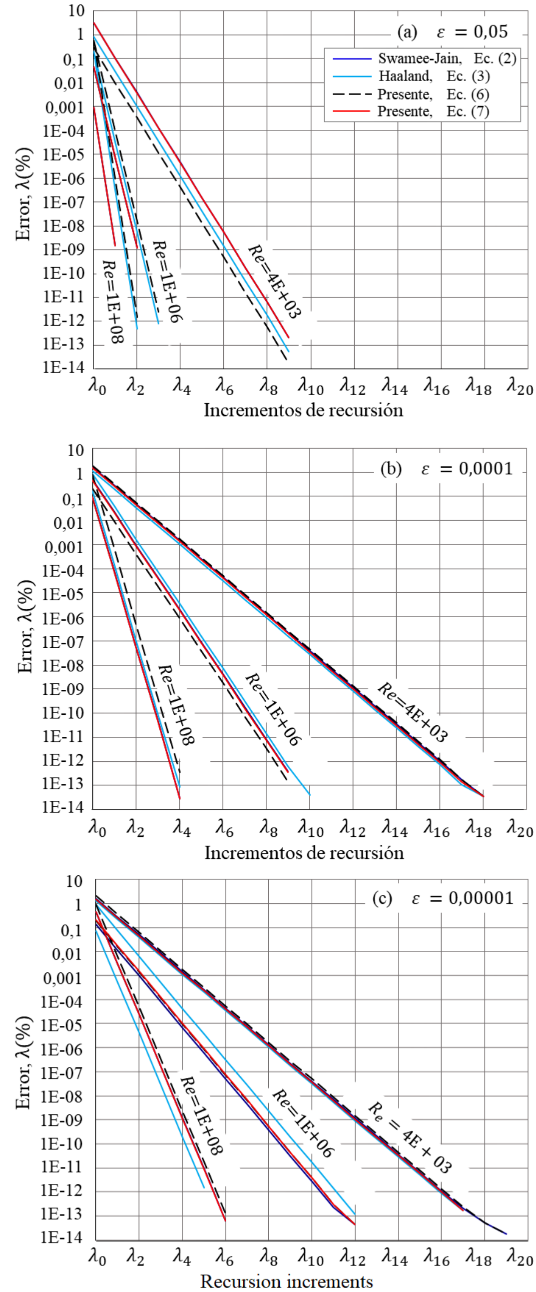

Figure 7. Percentage errors of the friction factor for equations (2), (3), (6) and (7)

Equation (9) is applicable for hydraulic pipes with values lower than $\varepsilon = 0,00001$, and even for $\varepsilon = 0$. When $\varepsilon = 0$, equation (9) simplifies, and numerically, the error curves of the friction factor resemble the trajectories shown in Figure 7c. The results from the graphs and tables are not presented due to their similarity. For λ_8, and the range $4000 \leq R_e \leq 1E + 08$, the percentage error is lower than 1,5E-06%.

The Swamee-Jain and Haaland equations were evaluated in equation (9), starting from λ_2, and yielded similar results to the evaluations of the proposed equations (6) and (7).

Any equation that has structures different from equations (6) and (7) and that is used in equation (9) can reduce the percentage error of the friction factor.

The advantage of equation (9) lies in its simple structure and ease of use for directly calculating the friction factor as an approximate solution.

4. Conclusions

The correlation expressed in equation (9) for λ_8, and the explicit relationships represented by equations (6) and (7) for calculating λ_o as an initial approximation provide an approximate solution for the implicit Colebrook-White equation (1). Equation (9) is applicable for turbulent flows within the Reynolds number range of $4000 \leq R_e \leq 1E + 08$ and the relative roughness range of $0,05 \geq \varepsilon \geq 0,00001$. It can also be applied to smooth pipes within the same Reynolds number range. However, equation (9) is not applicable for $R_e < 4000$.

Within the relative roughness range of, $0,05 \geq \varepsilon \geq 0,00001$, and Reynolds number range of $4 \times 10^{03} \leq R_e \leq 1 \times 10^{08}$, the estimated value of the maximum percentage error of the friction factor is $1,7 \times 10^{-06}$ %, for $\varepsilon = 0,00001$ and $R_e = 4 \times 10^{03}$. In this case, the friction factor has seven exact decimal digits. For other values of relative roughness and Reynolds numbers, the numerical magnitudes of the friction factor exhibit more than seven exact decimal digits.

For recursions beyond λ_8, the exact decimal digits increase. For $\varepsilon = 0,00001$ and $R_e = 4 \times 10^{03}$, λ_{10} exhibits an error of $4,95 \times 10^{-08}$ %, λ_{14} yields an error of $4,5 \times 10^{-11}$ %, and λ_{20} exhibits an error of 0,0%.

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